

Physical Metric and the Nature of Gravity

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Abstract

A physical metric is defined as one which gives a measurable speed of light throughout the whole space time continuum. It will be shown that a metric which satisfies the condition that speed of light on the spherical direction is that in a vacuum gives a correct result. All the metric functions thus obtained are positive definite and exhibits a repulsive force at short distances. The horizon in the sense of vanishing of the speed of light still exists in the radial direction. It is located at $3\sqrt{3}r_s/2 = 2.60 r_s$, where $r_s = 2GM/c^2$ is the Schwarzschild radius. This radius corresponds to the size of a black hole, as well as the photon sphere radius. The metric can be used to calculate general relativistic predictions in higher order for any process.

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I. INTRODUCTION

The Schwartzschild metric is the exact solution of the Einstein equation of general relativity[1],[2]. It is a metric for the spherically symmetric and static (SSS) system. However, since the speed of light in a spherical direction inside the horizon of the Schwarzschild metric is imaginary, it is not a physical metric. This characteristic, of course, is changed if the role of r and t is awitched inside the horizon. The speed of light becomes positive definite even inside the horizon. Then, the meaning of static is lost, i. e. the metric inside the horizon is non-static. In this article, the author constructs a physical metric by a coordinate transformation of the Schwarzschild metric, maintaining the characteristic of static nature and discusses the nature of gravity in the obtained physical metric. In a forthcoming article, the author discusses the difference between the Schwarzschild metric and the physical metric for the prediction of time delay experiment of Shapiro et.al[3].

II. ASYMPTOTIC FORM FOR THE PHYSICAL METRIC

The physical metric is expressed as

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - e^{\mu(r)} r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (1)$$

for a mass point M in a spherically symmetric and static (hereafter refered as SSS) metric. From the fact that the transformation, $r' = re^{\mu(r)/2}$, leads to the Schwarzschild metric[4], one can deduce the expression for the metric,

$$e^{\nu(r)} = 1 - (r_s/r)e^{-\mu(r)/2}, \quad (2)$$

$$e^{\lambda(r)} = \left(\frac{d}{dr} (re^{\mu(r)/2}) \right)^2 / (1 - (r_s/r)e^{-\mu(r)/2}), \quad (3)$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius. An asymptotic expansion for the metric functions can be obtained from Eq.(2) and Eq.(3), yielding

$$e^{\nu(r)} = \sum_{n=0}^{\infty} a_n (r_s/r)^n, \quad e^{\lambda(r)} = \sum_{n=0}^{\infty} b_n (r_s/r)^n, \quad \text{and} \quad e^{\mu(r)} = \sum_{n=0}^{\infty} c_n (r_s/r)^n, \quad (4)$$

where

$$a_0 = b_0 = c_0 = 1, \quad (5)$$

$$-a_1 = b_1 = 1 \quad \text{and} \quad (6)$$

$$a_2 = c_1/2, \quad b_2 = 1 - c_1/2 + c_1^2/4 - c_2, \quad \text{etc.} \quad (7)$$

It is obvious that a_{n+1} and b_n can be expressed as functions of $c_n, c_{n-1} \dots, c_1$.

III. CONSTRUCTION OF THE PHYSICAL METRIC IN THE ASYMPTOTIC REGION

Since the trouble of the Schwarzschild metric lies in the speed of light on a spherical direction inside the horizon, one can eliminate this trouble by requiring the following ansatz.

Proposition 1 *The speed of light in the angular direction in the SSS metric is the same as that of vacuum.*

In other words, we require

$$e^{\nu(r)} = e^{\mu(r)} = \omega \quad (8)$$

This ansatz implies that although gravity deforms the geometry of space-time, speed of light perpendicular to the gravity will not be affected.

Then one gets the equation for the asymptotic solution,

$$e^{\nu(r)} = 1 - (r_s/r)e^{-\mu(r)/2} = e^{\mu(r)} = \omega. \quad (9)$$

Then one has

$$r_s/r = e^{\mu(r)/2}(1 - e^{\mu(r)}) = \omega^{1/2}(1 - \omega), \quad (10)$$

or

$$(r_s/r)^2 = \omega(1 - \omega)^2. \quad (11)$$

Differentiating Eq.(11), one gets

$$r \frac{d\omega}{dr} = \frac{2r_s^2/r^2}{(1 - \omega)(3\omega - 1)} = \frac{2\omega(1 - \omega)}{(3\omega - 1)}. \quad (12)$$

From Eq.(3), the metric function in the radial direction can be calculated

$$e^{\lambda(r)} = \left(\frac{d}{dr}(r\omega^{1/2})\right)^2/\omega = (\omega^{1/2} + \omega^{-1/2}r \frac{d\omega}{dr}/2)^2/\omega = \left(\frac{2\omega}{3\omega - 1}\right)^2. \quad (13)$$

From Eq.(10) or Eq.(11) and Fig. 1, it is clear that one covers the range of

$$1 > \omega > 1/3 \quad (14)$$

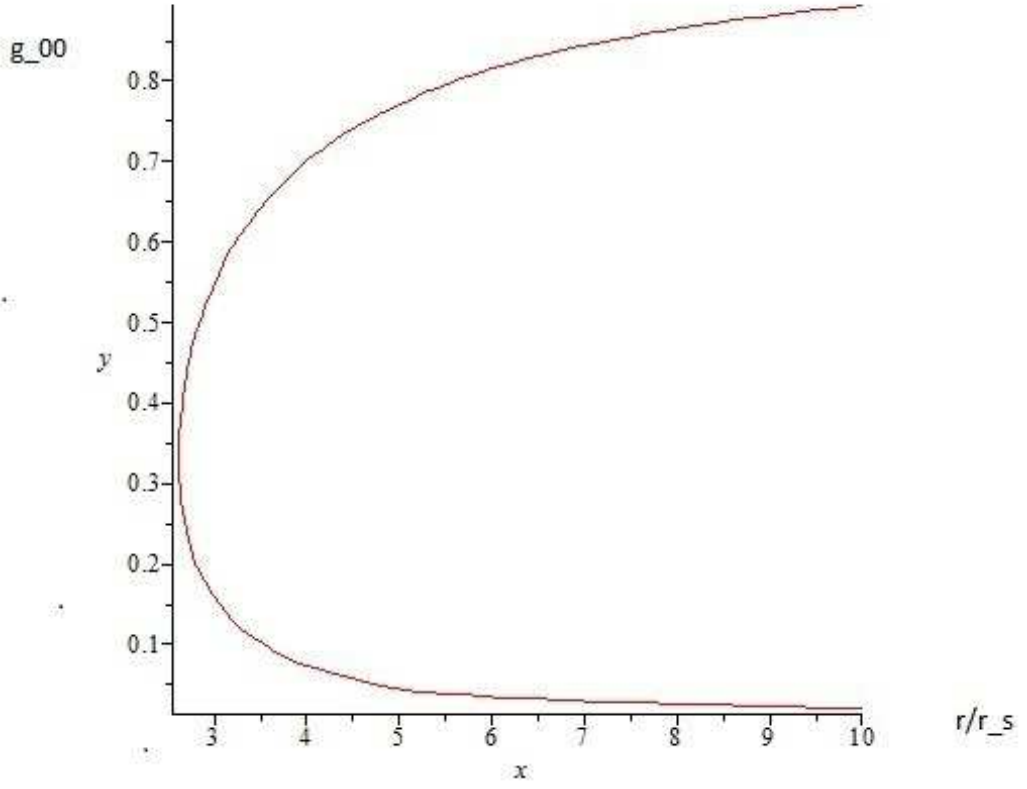


Fig. 1

and

$$\infty > r/r_s > 3\sqrt{3}/2. \quad (15)$$

In order to cover the range of

$$r/r_s < 3\sqrt{3}/2, \quad (16)$$

one has to use non-asymptotic solution of the Schwarzschild solution. From Appendix, such a solution is given in the next section.

The asymptotic expansion of the metric functions can be calculated from Eq.(11) and Eq.(13) as

$$\omega = e^{\nu(r)} = e^{\mu(r)} = 1 - (r_s/r) - \frac{1}{2}(r_s/r)^2 - \frac{5}{8}(r_s/r)^3 - (r_s/r)^4 - \dots \quad (17)$$

and

$$e^{\lambda(r)} = 1 + (r_s/r) + \frac{9}{4}(r_s/r)^2 + \frac{43}{8}(r_s/r)^3 + \frac{211}{16}(r_s/r)^4 + \dots \quad (18)$$

where Eq.(13) has been used. Successive expansion yields a determination of all the parameters, c_n , for the physical metric. These are useful for testing observational data in higher order in gravity. Alternatively, the inverse function of Eq.(10) or Eq.(11) may be used.

IV. THE PHYSICAL METRIC IN THE WHOLE REGION

The Schwarzschild solution for non-asymptotic region (See Appendix section) can be written as

$$e^{\lambda(r)} = (1 + \frac{Dr_s}{r'})^{=1} \quad (19)$$

and

$$e^{\nu(r)} = \frac{1}{A}(1 + \frac{Dr_s}{r'}), \quad (20)$$

where A and D are non-dimensional constants. The metric functions for the physical metric in the region

$$r/r_s < 3\sqrt{3}/2 \quad (21)$$

are expressed as

$$\omega = e^{\nu(r)} = \frac{1}{A}(1 + D(r_s/r)e^{-\mu(r)/2}) = e^{\mu(r)} \quad (22)$$

and

$$e^{\lambda(r)} = (\frac{d}{dr}(r\omega^{1/2}))^2/A\omega, \quad (23)$$

and hence

$$D(\frac{r_s}{r}) = \omega^{1/2}(A\omega - 1) \quad (24)$$

or

$$(\frac{Dr_s}{r})^2 = \omega(A\omega - 1)^2 \quad (25)$$

Differentiating Eq.(24), one gets

$$r \frac{d\omega}{dr} = -2\omega \frac{A\omega - 1}{3A\omega - 1} \quad (26)$$

and

$$e^{\lambda(r)} = (\omega^{1/2} + r \frac{d\omega}{dr} / 2\omega^{1/2})^2 / A\omega = A(\frac{2\omega}{3A\omega - 1})^2 \quad (27)$$

Imposing the continuity of the asymptotic expression, Eq.(10) and the non-asymptotic expression, Eq.(24) at

$$(r/r_s, \omega) = (3\sqrt{3}/2, 1/3) \quad (28)$$

one gets

$$A = 2D + 3. \quad (29)$$

One splits the parameter space in the following 3 Regions.

- I) $A > 3$ and $D > 0$,
- II) $3 > A > 0$ and $0 > D > -\frac{3}{2}$
- III) $0 > A$ and $-\frac{3}{2} > D$.

In Region I, the distance r can be reached at zero when ω reaches ∞ , as

$$\omega = (\frac{Dr_s}{Ar})^{2/3} \quad (30)$$

In Region II, the distance r cannot reach at zero value (the origin). In Region III, the distance r can be reached at zero, as in Eq.(30). However, the radial metric function, $e^{\lambda(r)}$, becomes negative as is seen from Eq.(23) or Eq.(27). This means that the radial velocity of light becomes imaginary. Therefore the most natural choice for the parameter space is the Region I, where all metric functions are positive definite. This is a characteristic of the physical metric.

V. THE NATURE OF THE PHYSICAL METRIC

Summarizing the previous sections, one obtains the physical metric

$$e^{\nu(r)} = e^{\mu(r)} = \omega \quad (31)$$

in the following manner. For the asymptotic region

$$r/r_s \geq 3\sqrt{3}/2, \quad (32)$$

one has

$$r_s/r = \omega^{1/2}(1 - \omega) \quad (33)$$

and

$$e^{\lambda(r)} = \left(\frac{2\omega}{3\omega - 1}\right)^2. \quad (34)$$

The range of ω is restricted to

$$\omega \geq 1/3. \quad (35)$$

For the non-asymptotic region

$$3\sqrt{3}/2 \geq r/r_s > 0, \quad (36)$$

one has

$$D\left(\frac{r_s}{r}\right) = \omega^{1/2}(A\omega - 1) \quad (37)$$

and

$$e^{\lambda(r)} = A\left(\frac{2\omega}{3A\omega - 1}\right)^2. \quad (38)$$

The continuity of Eq.(33) and Eq.(37) at

$$(r_s/r, \omega) = (3\sqrt{3}/2, 1/3) \quad (39)$$

requires

$$A = D/2 + 3. \quad (40)$$

Choosing the range of the parameter space to be

$$D > 0, \quad \text{and} \quad A > 3, \quad (41)$$

all the metric functions are positive definite, which guarantees the condition for the physical metric, having the definite value for speed of light throughout the whole space-time continuum.

At the origin, the metric function, $\omega = g_{00}(r)$, diverges as

$$\omega = \left(\frac{Dr_s}{Ar}\right)^{2/3}. \quad (42)$$

From Eq.(34), one concludes that $e^{\lambda(r)} = g_{11}(r)$ becomes ∞ at the edge point of the asymptotic region, Eq.(39). Hence, at $r_s/r = 3\sqrt{3}/2$, the radial speed of light vanishes. One may call this point a horizon, although the characteristics are very different from that of the Schwarzschild metric. The speed of light in the spherical direction is that of vacuum. Since all metric are positive definite in the whole region, speed of light is well defined throughout the whole space time. The magnitude of the horizon is $(3\sqrt{3}/2) r_s = 2.60 r_s$, i.e., 2.6 times

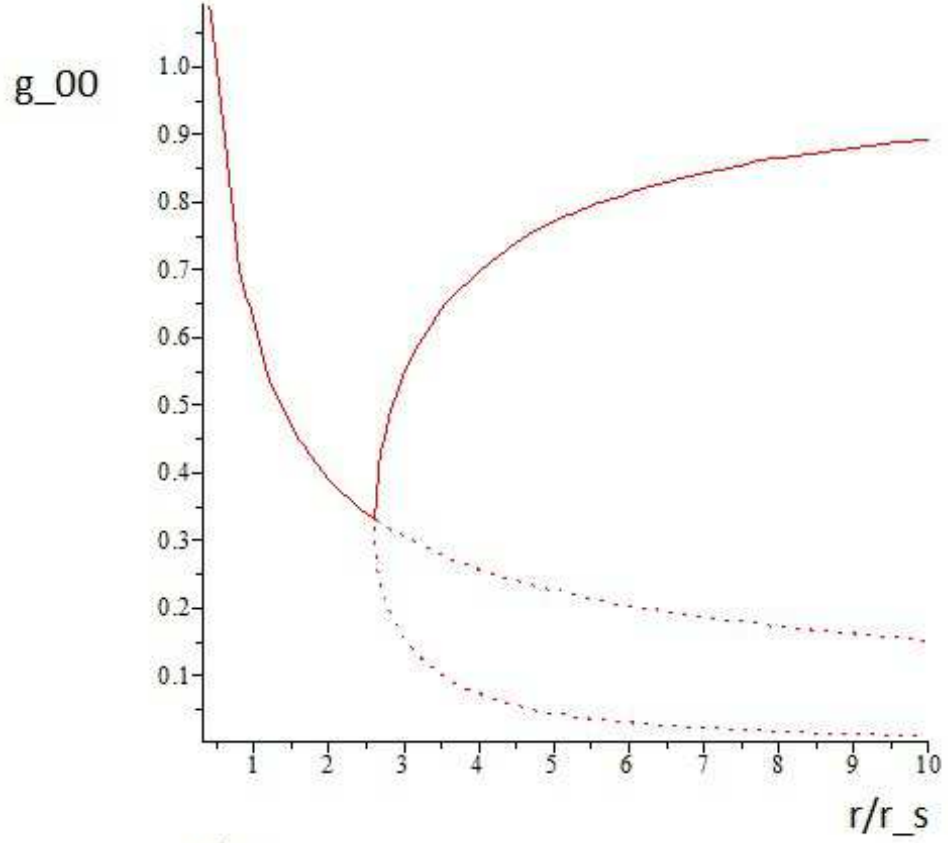


Fig.2

bigger than that of the Schwartzschild radius. Below Fig. 2 shows the picture of $g_{00}(r) = e^{\nu(r)} = \omega$ as a function of r/r_s , namely the picture of the gravitational potential with the shift of the y axis and a scale factor of 2.

The crossing point of the two curves is

$$(r/r_s, \omega) = (3\sqrt{3}/2, 1/3). \quad (43)$$

The radial metric function, $g_{11} = e^{\lambda(r)}$, is inevitably discontinuous at this point. This discontinuity allows the passing of all particles through the horizon, in and out. More

importantly, the gravitational potential inside the horizon is repulsive. This property could change the nature of gravity, black holes, cosmic ray production as well as the nature of cosmology. I will discuss these problems in the forthcoming articles.

VI. SUMMARY

The author has constructed a physical metric for which the speed of light is well defined throughout all space time continuum. The constructed metric shows that gravity is repulsive inside the horizon and the size of black holes is 3 times bigger than the Schwarzschild radius. The latter can be tested by observation in the near future[5]. The observational effects of the physical metric will be discussed in forthcoming articles.

VII. APPENDIX. THE SCHWARZSCHILD SOLUTION

Setting

$$e^{\mu(r)} = 1, \quad (44)$$

in Eq.(1), and using the Maple program the Einstein equation reads

$$-r\lambda'(r) - e^{\lambda(r)} + 1 = 0, \quad (45)$$

$$-r\nu'(r) + e^{\lambda(r)} - 1 = 0 \quad (46)$$

and

$$2\nu'(r) - 2\lambda'(r) + 2r\nu''(r) + r\nu'(r)^2 - r\nu'(r)\lambda'(r) = 0. \quad (47)$$

From the sum of Eq.(45) and Eq.(46), one gets

$$\nu'(r) + \lambda'(r) = 0. \quad (48)$$

Using this relation, Eq.(47) becomes

$$-r\lambda''(r) + r\lambda'(r)^2 - 2\lambda'(r) = 0 \quad (49)$$

or equivalently

$$e^{\lambda(r)}(re^{-\lambda(r)})'' = 0. \quad (50)$$

On the other hand, Eq.(45) can be written as

$$(re^{-\lambda(r)})' = 1, \quad (51)$$

which solution is

$$e^{-\lambda(r)} = 1 + \frac{B}{r}, \quad (52)$$

and Eq.(50) is satisfied, where B is an integration constant. The solution of Eq.(48) reads

$$e^{\nu(r)} = \frac{1}{A} \left(1 + \frac{B}{r}\right). \quad (53)$$

The asymptotic solution with the boundary condition is given by

$$A = 1, \quad B = -r_s. \quad (54)$$

On the other hand, the non-asymptotic solution is given by

$$B = Dr_s, \quad A \text{ arbitrary}. \quad (55)$$

where A and D are nondimensional integration constants.

Acknowledgments

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Figure captions

Fig. 1 The metric function, $g_{00}(r)$, in the asymptotic region in the SSS physical metric.

Fig. 2 The metric function, $g_{00}(r)$, as a function of r/r_s in the SSS physical metric.